Nonsupersymmetric D-Branes and the Kitaev Fermion Chain

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Supersymmetric D-branes have been important in string theory since Polchinski (1994) and a few years later A. Sen showed that nonsupersymmetric D-branes also play a role.

These nonsupersymmetric D-branes have subtle properties that were nicely elucidated by Sen. (I should also mention work that was done a little later by Kraus and Larsen, which was somewhat in the spirit of what I will say.)
Because time will be short, I will only explain the simple case in which there is no time-reversal symmetry and the anomaly associated to the Kitaev chain is a mod 2 effect. That means we will be considering Type II superstrings. With time-reversal included, we would have Type I superstrings and a mod 8 anomaly, just as in the T-invariant version of Kitaev’s chain. The string theory applications are more significant in the T-invariant case (as Sen showed) but today I will only describe the basic picture without T symmetry.
The story will revolve around a certain mod 2 anomaly, whose most simple manifestation is in spacetime dimension 1 (that is, 0 space and 1 time dimensions).
Any compact 1-manifold is a circle:

But there are two spin structures, as a fermion field $\lambda$ may be periodic or antiperiodic in going around the circle. In string theory these are called Ramond and Neveu-Schwarz spin structures.
The path integral with antiperiodic fermions (NS spin structure) on a circle of circumference $\beta$, in a theory with Hamiltonian $H$, computes

$$\text{Tr} \exp(-\beta H)$$

and the path integral with periodic fermions (R spin structure) computes

$$\text{Tr} (-1)^F \exp(-\beta H).$$
In the periodic or Ramond spin structure, $\lambda$ has a single zero-mode. These means that the path integral is trying to produce a nonzero value of

$$\langle \lambda \rangle.$$

Such a nonzero value violates the symmetry $(-1)^F$, which is fundamental in all areas of physics. Thus it is inherently an anomaly. Related to this, though $\lambda$ has a 1-dimensional space of zero-modes, there is no natural choice of sign for this mode and hence there is no way to make sense of the sign of the nonzero matrix element $\langle \lambda \rangle$. Any procedure that would tell us what the sign should be would explicitly violate the $(-1)^F$ symmetry.
By contrast, in the NS spin structure, there is no problem with the path integral on a circle:

![Circle Diagram]

Its numerical value is $\sqrt{2}$. 
Notice that when we take an even number of copies of the original theory, then in the Ramond spin structure we have an even number of zero-modes, so there is no anomaly in \((-1)^F\). So this is a mod 2 anomaly and it is our first example of a mod 2 anomaly.
In this audience, probably most of you know one thing we can do with this anomaly: we consider the $0+1$-dimensional system to exist on the boundary of a $1+1$-dimensional world, on which there is a gapped but topologically nontrivial system – the Kitaev fermion chain. However, I will describe that system in a somewhat abstract, relativistic language.
To describe fermions on a two-manifold that might be curved or topologically non-trivial,

we need a “spin structure” which tells us the sign that we get when a spin 1/2 field is parallel-transported around a noncontractible loop. For example, in the case of a torus, there are $2 \times 2 = 4$ spin structures.
On an (orientable) two-manifold with a spin structure, there is an important mod 2 invariant that I will call $\zeta$. To define it, we consider a positive chirality fermion $\chi$ on $\Sigma$. The Dirac action

$$\int_\Sigma d^2x \sqrt{g} \bar{\chi} D\chi$$

is antisymmetric, by fermi statistics.
The canonical form of an antisymmetric matrix is

\[
\begin{pmatrix}
0 & -a \\
a & 0 \\
0 & -b \\
b & 0 \\
0 & 0 & \ddots
\end{pmatrix}
\]

with nonzero modes that come in pairs and zero modes that are not necessarily paired. The number of zero modes can change only when one of the “skew eigenvalues” \( a, b, \cdots \) becomes zero or nonzero, and when this happens, the number of zero-modes jumps by 2. So the number of zero-modes mod 2 is a topological invariant, called the mod 2 index. I will denote it as \( \zeta \). It is called the mod 2 index of the (chiral) Dirac operator.
As an example, consider a Riemann surface of genus 1:

![Diagram of a Riemann surface of genus 1]

There are four spin structures, usually labeled $\pm\pm$. The $++$ spin structure has a single positive chirality zero-mode (the “constant” mode of $\chi$) and the other spin structures have none. So the $++$ spin structure is odd, with $\zeta = 1$, and the others are even, with $\zeta = 0$. 
There is a fermionic SPT phase with no symmetry except \((-1)^F\) in which the partition function on an oriented two-manifold is \((-1)^\zeta\). However, on a manifold with boundary, the quantity \(\zeta\) cannot be defined as a topological invariant, because there is no boundary condition that would be suitable. So there is a subtlety in defining this SPT phase on a manifold with boundary.
However, the SPT phase with partition function \((-1)^\zeta\) does make sense on a two-manifold with boundary if on the boundary one has the anomalous \(0 + 1\)-dimensional theory we started with:

\[
\begin{array}{c}
\text{In other words, let } \lambda \text{ be a real fermion on the boundary and } D \text{ the corresponding } 0 + 1\text{-dimensional Dirac operator. The path integral of a real fermion is formally a Pfaffian } \text{Pf}(D) \text{ (this is roughly the same as } \sqrt{\det(D)})\text{. Though Pf}(D) \text{ and } (-1)^\zeta \text{ do not make sense separately in this situation, one can define the product:}
\end{array}
\]

\[
Pf(D)(-1)^\zeta.
\]
One way to see that this must be true is to construct an explicit gapped system with partition function \((-1)^\zeta\), and see how it behaves on a manifold with boundary. The most obvious gapped system of fermions in two dimensions with no extra symmetry assumed is a massive Majorana fermion \(\psi\) with Dirac equation \((\slashed{D} + m)\psi = 0\). Formally its path integral is \(\text{Pf}(\slashed{D} + m)\). This is real (because the Dirac operator is real) but we should ask if there is a subtlety in defining its sign. It is best for the sort of argument that I will give to make sure that the starting point is completely well-defined and anomaly free. To do this, we can start with two identical Majorana fermions \(\psi_1, \psi_2\) with the same mass \(m\). Then the path integral is \((\text{Pf}(\slashed{D} + m))^2\). This is the square of a real number, so it is positive and thus completely anomaly-free. Because the system is gapped, its path integral \((\text{Pf}(\slashed{D}_m))^2\) is completely trivial at long distances, except for nonuniversal terms that can be removed by counterterms.
Now to get something more interesting, let us flip the sign of the mass of one of the two fermions, say $\psi_1$. So now the mass parameter in the Dirac operator is $-m$ for $\psi_1$ and $+m$ for $\psi_2$. It does not matter if we consider the real parameter $m$ to be positive or negative. To get from one theory to the other, we can make a chiral rotation $\psi_1 \rightarrow \bar{\gamma}\psi_1$, where $\bar{\gamma} = i\gamma_1\gamma_2$ is the chirality operator. This acts as $+1$ on modes of $\psi_1$ of positive chirality, and as $-1$ on modes of negative chirality. So it transforms the path integral measure by a factor of $-1$ for every negative chirality mode of $\psi_1$. The nonzero modes come in pairs (because of a variant of Kramers doubling) and do not contribute any anomaly. But the number of negative (or positive) chirality zero-modes is by definition equal to $\zeta$ mod 2. So under $\psi_1 \rightarrow \bar{\gamma}\psi_1$, the path integral measure for the zero-modes picks up a factor $(-1)^\zeta$. 
Thus the theory with two otherwise identical Majorana fermions with opposite signs of the mass (but regulated in the UV in the same way) gives a physical realization of a theory in the SPT phase, with partition function $(-1)^ζ$. Now let us consider what happens near the boundary:
We need a boundary condition. There are two equally good boundary conditions, namely \( n\psi^1 = +\psi \) and \( n\psi^2 = -\psi \), where \( \vec{n} \) is the normal vector to the boundary and \( n = \vec{\gamma} \cdot \vec{n} \). To make sure the boundary condition introduces no anomaly, we use the same boundary condition for both \( \psi^1 \) and \( \psi^2 \). Thus if they have the same mass \( m \), we still have a completely trivial theory, its path integral being the square of a real number. Now consider the interesting case that they have equal and opposite masses \( m \) and \( -m \). (Note that anomalies never depend on masses, so changing the masses but leaving the boundary conditions alone will not introduce an anomaly.)
The Dirac equation is

\[ 0 = (\not{D} + m)\psi = (\gamma^1 D_1 + \gamma^2 D_2 + m)\psi. \]

Consider this equation on a half-space \( x^1 \geq 0 \)

and look for a zero-mode that is localized along the boundary.
For a mode independent of \( x^2 \), and on a flat half-space, the equation reduces to

\[
\frac{\partial \psi}{\partial x} = -m \gamma_1 \psi,
\]

where I set \( x = x^1 \). So

\[
\psi(x) = \exp(-m \gamma_1 x) \psi(0).
\]

If the boundary condition is

\[
\gamma_1 \psi(0) = \varepsilon \psi(0), \quad \varepsilon = \pm 1,
\]

then

\[
\psi(x) = \exp(-m \varepsilon x) \psi(0).
\]

This is localized along the boundary \( x = 0 \) if and only if \( m \varepsilon > 0 \).
For the trivial theory, both $\psi_1$ and $\psi_2$ have the same sign of $m$, so the number of boundary-localized modes is either 0 or 2, an anomaly-free combination. The bulk theory is trivial and the boundary theory is anomaly-free. For the nontrivial theory, $\psi_1$ and $\psi_2$ have opposite signs of the mass, so for either choice of $\varepsilon$, one of them has a boundary-localized mode and one does not. Thus if the bulk theory is nontrivial, there will be a single edge-localized Majorana mode on the boundary.
There is a small variant of this that might be less familiar. Let us now consider the *trivial* theory, with $\psi_1$ and $\psi_2$ having the same sign of $m$. But let us flip the sign of the boundary condition for $\psi_1$ relative to $\psi_2$, so one of them has $\varepsilon = +1$ and one has $\varepsilon = -1$. We are *not* guaranteed by any general consideration that this setup will be anomaly-free, and in fact it is anomalous. To see this, we just note that flipping the boundary condition for one of the two fields will lead to a setup in which $\psi_1$ or $\psi_2$ but not both has a boundary-localized mode. So *flipping the boundary condition of one fermion contributes to the anomaly.*
This last statement might be slightly unfamiliar, but actually it is closely related to mean field theories of the Kondo effect.

For the application to string theory, we need the following: since anomalies do not depend on masses, flipping the boundary condition of one fermion contributes to the anomaly whether the fermion has a bare mass or not.
Let us define a discrete theta angle $\alpha$, which is either 0 or 1, by saying that the bulk partition function is $(-1)^{\alpha \zeta}$. And let $n$ be the number of boundary-localized fermions, and $r$ the number of bulk fermions with $\varepsilon = -1$. We have a general condition for anomaly-cancellation:

$$n + r + \alpha \cong 0 \mod 2.$$
Now let us turn to string theory. String perturbation theory is defined by a path integral on a two-manifold, the worldsheet $\Sigma$ of the string. This path integral describes a map from $\Sigma$ to a spacetime manifold $M$. In addition to bosons that describe that map, the worldsheet path integral has 10 Majorana fermions related to the bosons by worldsheet supersymmetry. There are two string theories, the Type IIA and Type IIB superstring theory, in which the worldsheet is oriented (there is no $T$ symmetry) and our discussion precisely applies: their worldsheet path integrals differ precisely by a factor of $(-1)^\zeta$, which is present for Type IIA and not for Type IIB.
Let $M$ be spacetime. To describe a “brane,” which is a submanifold $N$ of spacetime on which strings are allowed to end, we place a constraint that says that while the string worldsheet is mapped to $M$, its boundary is mapped to $N$:

If $N$ is of codimension $r$ in $M$, then worldsheet supersymmetry forces us to flip the boundary condition on $r$ of the worldsheet fermions.
Polchinski (1994) constructed branes with spacetime supersymmetry. These branes had $r$ even for Type IIB superstring theory and $r$ odd for Type IIA. We can unify this by saying that Polchinski constructed all the branes with

$$r + \alpha = 0$$

where $\alpha = 0$ for Type IIB and $\alpha = 1$ for Type IIA. The anomaly that we’ve been discussing was not part of Polchinski’s reasoning, but we can see that in fact, he constructed all of the anomaly-free branes that do not have edge-localized fermions.
Sen (ca. 1998) constructed “nonsupersymmetric branes” with $r + \alpha = 1$. Again his considerations did not involve anomalies, at least not in a direct fashion. Nor was his answer described in terms of a theory with an edge-localized fermion. But from our discussion, we can see how to construct an anomaly-free theory with $r + \alpha = 1$. We simply add a boundary-localized fermion, so $n = 1$. Thus we can get all values of $r$ for both Type IIA and Type IIB as long as we respect the anomaly-cancellation condition

$$n + r + \alpha = 0 \pmod{2}.$$
Sen gave a clear and consistent, but slightly baroque, set of rules for calculations in the presence of a nonsupersymmetric brane. His rules have simple explanations in the new language. For example, when a string ends on a nonsupersymmetric brane, the space of operators that can be inserted on the string endpoint is according to Sen roughly twice as big as for a supersymmetric brane.

It is now clear why: such an operator might be an operator $V$ constructed from usual fields on the string or it might be $\lambda V$ where $\lambda$ is the edge-localized fermion.
As I said at the outset, there is a richer version of all this with $T$ symmetry, in Type I superstring theory. Sen’s most interesting results were actually in that context. But we will have to leave that for another time.